PRINCIPLES OF ANALYSIS
LECTURE 19 - INTERMEDIATE VALUE THEOREM

PAUL L. BAILEY

1. Relatively Open Sets

Let $E \subset \mathbb{R}$ and let $V \subset E$. We say that E is *relatively open* in E if there exists an open set $U \subset \mathbb{R}$ such that $U \cap E = V$. Similarly, a subset $G \subset E$ is *relatively closed* if $E \smallsetminus G$ is relatively open. This is equivalent to the existence of a closed set $F \subset \mathbb{R}$ such that $F \cap E = G$.

Proposition 1. *Let $f : E \to \mathbb{R}$ be a function. Then f is continuous on E if and only if for every open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is relatively open in E.*

2. Homeomorphism

Let A and B be subsets of $\mathbb{R}$. A *homeomorphism* from A to B is a bijective continuous function $f : A \to B$ such that f^{-1} is also continuous.

It is natural to suppose that any bijective continuous function is a homeomorphism, but this is not the case.

Example 1. Let $A = (0, 1) \cup [2, 3)$ and let $B = (0, 2)$. Define $f : A \to B$ by

$$f(x) = \begin{cases} x & \text{if } x \in (0, 1); \\ x - 1 & \text{if } x \in [2, 3). \end{cases}$$

This function is clearly bijective and continuous at every point in A; however, its inverse is discontinuous.

We have seen that the continuous image of a compact set is compact. We will use this fact in the next proposition.

Date: November 5, 2003.

Proposition 2. *Let $f : A \to B$ be a bijective continous function. If A is compact, then f is a homeomorphism.*

Lemma 1. *If F is closed and U is open, then $F \smallsetminus U$ is closed and $U \smallsetminus F$ is open.*

Proof of Lemma. Since $F \smallsetminus U = F \cap (\mathbb{R} \smallsetminus U)$ is the intersection of closed sets, it is closed. On the other hand, since $U \smallsetminus F = U \cap (\mathbb{R} \smallsetminus F)$ is the intersection of open sets, it is open. □

Proof of Proposition. Let $g = f^{-1}$ so that $g : B \to A$ is a bijective function; we wish to show that g is continuous.

Let $\epsilon > 0$ and select $x_0 \in B$. Since A is compact, it is closed and bounded Let $U = (g(x_0) - \epsilon, g(x_0) + \epsilon)$. Then U is open, and $K = A \smallsetminus U$ is also closed and bounded, and hence compact. Since the continuous image of a compact set is compact, we see that $f(K)$ is compact, and hence closed. Let $V = \mathbb{R} \smallsetminus f(K)$; this set is open. Note that

$$\begin{aligned} g(B \cap V) &= g(B \smallsetminus f(K)) \\ &= g(B) \smallsetminus g(f(K)) \\ &= g(B) \smallsetminus K \\ &= A \smallsetminus (A \smallsetminus U) \\ &= U. \end{aligned}$$

Now $g(x_0) \notin K$, so $x_0 = f(g(x_0)) \notin f(K)$, so $x_0 \in V$. Therefore there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset V$. Thus if $x \in B$ and $|x - x_0| < \delta$, we have $f(x) \in U$, which says that $|f(x) - f(x_0)| < \epsilon$. □

3. Connectedness Revisited

Recall the definition of a closed interval:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Recall the definition of connectedness:

A subset $A \subset \mathbb{R}$ is *disconnected* if there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ with $A \cap U_1 \neq \varnothing$ and $A \cap U_2 \neq \varnothing$ such that $A \subset (U_1 \cup U_2)$. Otherwise, we say that A is *connected.*

Proposition 3. *Let $f : E \to \mathbb{R}$ be a continuous function, If E is connected, then $f(E)$ is connected.*

Proof. It suffices to show that if $f(E)$ is disconnected, then E is disconnected. Thus assume that $f(E)$ is disconnected, and let V_1 and V_2 be open subsets of $\mathbb{R}$ such that $f(E) \cap V_1 \neq \varnothing$, $f(E) \cap V_2 \neq \varnothing$, but $f(E) \subset (V_1 \cup V_2)$.

Let $E_1 = f^{-1}(V_1)$ and $E_2 = f^{-1}(V_2)$. We wish to find disjoint open sets U_1 and U_2 such that $E_1 = E \cap U_1$ and $E_2 = E \cap U_2$.

For each $y \in f(E)$ there exists $\epsilon_y > 0$ such that $(y - \epsilon_y, y + \epsilon_y) \subset V_i$, where $y \in V_i$. Since f is continuous, for each $x \in E$ there exists $\delta_x > 0$ such that $f((x - \delta_x, x + \delta_x)) \subset (y - \epsilon_y, y + \epsilon_y)$, where $y = f(x)$.

Set $U_i = \cup_{x \in E_i} (x - \delta_x, x + \delta_x)$, for $i = 1, 2$. Then U_1 and U_2 are open sets. Also $E \cap U_1 \neq \varnothing$, $E \cap U_2 \neq \varnothing$, but $E \subset (U_1 \cup U_2)$. Thus, E is disconnected. □

Proposition 4. *Let $A \subset \mathbb{R}$. Then A is connected if and only if*

$$a, b \in A \Rightarrow [a, b] \subset A.$$

Proof. We prove both directions.

($\Rightarrow$) Let $a, b \in A$ with $a < b$ and suppose that $[a, b]$ is not contained in A. Then there exists $c \in [a, b]$ such that $c \notin A$. Set $U_1 = (-\infty, c)$ and $U_2 = (c, \infty)$; then $a \in U_1$, $b \in U_2$, and $A \subset U_1 \cup U_2$. Thus A is disconnected.

($\Leftarrow$) Suppose that for every $a, b \in A$ with $a < b$, we have $[a, b] \subset A$. Let U_1 and U_2 be open sets with $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. We wish to show that $U_1 \cap U_2 \neq \varnothing$.

Let $a \in U_1$ and $b \in U_2$; without loss of generality, assume that $a < b$. Let $c = \sup U_1 \cap [a, b]$. Clearly $c \in [a, b]$, so either $c \in U_1$ or $c \in U_2$.

If $c \in U_1$, then there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_1$. Thus $c + \min\{\frac{\epsilon}{2}, \frac{c+b}{2}\}$ is also in U_1 and in $[a, b]$, contradicting the definition of c.

Thus $c \in U_2$, so there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_2$. But by the definition of c, there exists $d \in U_1 \cap [a, b]$ such that $d \in (c - \epsilon, c) \subset U_2$. Thus $U_1 \cap U_2 \neq \varnothing$. □

Proposition 5. *Let $K \subset \mathbb{R}$ be a compact set. Then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Exercise. □

Proposition 6. *A compact connected set is a closed interval.*

Proof. Exercise. □

4. Intermediate Value Theorem

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) \cdot f(b) < 0$. Then there exists $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Since $f([a, b])$ is connected, $[f(a), f(b)] \subset f([a, b])$. Since $0 \in [f(a), f(b)]$, then $0 \in f([a, b])$. That is, $f(c) = 0$ for some $c \in f([a, b])$. □

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu